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On the approximation of time one maps of Anosov flows by Axiom A diffeomorphisms

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Abstract. We prove that if f_1 is the time one map of a transitive and codimension one Anosov flow ϕ and it is C^1-approximated by Axiom A diffeomorphisms satisfying a property called $\mathcal{P}$, then the flow is topologically conjugated to the suspension of a codimension one Anosov diffeomorphism.
A diffeomorphism f satisfies property $\mathcal{P}$ if for every periodic point in M the number of periodic points in a fundamental domain of its central manifold is constant.

Keywords: Anosov flow, suspension of Anosov diffeomorphisms, time one map, Axiom A.

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Introduction

Throughout this paper M denotes a smooth compact Riemannian manifold without boundary, and $\phi : M \times \mathbb{R} \to M$ a C^r flow, with $r \geq 1$.

Recall that the suspension of an Anosov diffeomorphism is an Anosov flow in the corresponding manifold. Let us consider a transitive Anosov vector field X and let $f_\tau = X_\tau$ be the flow of X at time τ. Although f_τ is not an Anosov diffeomorphism, there exists a Df_τ-invariant splitting of TM

$$TM = E^s \oplus E^c \oplus E^u,$$

such that $Df_\tau|E^s$ is uniformly contracting, $Df_\tau|E^u$ is uniformly expanding, and E^c is a nonhyperbolic central direction.

The object of our study are *transitive* Anosov flows (i.e. the case when the non-wandering set is the whole manifold).
A codimension one Anosov flow defined on an n-manifold M is an Anosov flow

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such that for all $x \in M$, $dim E^s(x) = 1$ and $dim E^u(x) = n-2$ or $dim E^s(x) = n-2$ and $dim E^u(x) = 1$.

An interesting question is what kind of dynamical system can appear under perturbations of a time one map of a transitive Anosov flow.

Palis and Pugh (see [9]) wondered whether the time one map of a transitive Anosov flow could be approximated by hyperbolic or Axiom A diffeomorphisms. It is a well known fact that in the case when the flow arises from the suspension of an Anosov diffeomorphism $g : N \to N$ such an approximation can be carried out with Axiom A diffeomorphisms.

The suspension manifold N_g is obtained from the direct product $N \times [0, 1]$ by identifying pairs of points of the form $(x, 0)$ and $(g(x), 1)$ for $x \in N$. The suspension flow $\varphi(x, t)$ is determined by the vector field $\frac{\partial}{\partial t}$. The manifold N_g is fibered over S^1 and the projection of $\varphi(x, 1)$ onto S^1 is the identity map. Let f be a diffeomorphism preserving fibers, C^1- close to $\varphi(x, 1)$ such that the projection of f over S^1 is a Morse-Smale map. We have that f is an Axiom A diffeomorphism.

In this spirit, Bonatti and Díaz (see [2]) proved that if τ is a period of a periodic orbit of a transitive Anosov flow, then there exist an open set $\mathcal{U}$ of nonhyperbolic and transitive diffeomorphisms, and a sequence $(g_n)_{n\in\mathbb{N}}$, $g_n \in \mathcal{U}$ such that $g_n \to f_\tau$ in the C^1- topology.

Throughout this paper τ will be 1.

Our aim is to give a partial answer to the Palis-Pugh question. We will say that a diffeomorphism f satisfies property $\mathcal{P}$ if for any periodic point x the number of periodic points between x and $f(x)$ in the connected component of its central manifold is constant (see Section 1).

This property is not so strange. It is, for instance, verified in the case when f is a convenient C^1-perturbation of the time one map of a transitive Anosov flow arising from the suspension of an Anosov diffeomorphism. In fact, the above example verifies that the number of periodic points between x and $f(x)$ in the connected component of its central manifold is constant, if x is a periodic point of f.

We will show that, in the general case there is an open dense set $V \subset M$ such that the number of periodic points between x and $f(x)$ is constant for all the f-periodic points in V. Here, as before, f is a C^1-perturbation of the time one map of a transitive Anosov flow.

We will prove the following:

Theorem 1. *Let M a smooth compact riemannian manifold without boundary, $dim(M) \geq 3$. If the time one map of a transitive codimension one Anosov flow is C^1-approximated by Axiom A diffeomorphisms satisfying property $\mathcal{P}$, then the flow is topologically conjugated to a suspension of a codimension one Anosov diffeomorphism.*

Perhaps it is worthwhile to note that Verjovsky (see [10]) proved that if $n > 3$ any codimension one Anosov flow is transitive (see [5] for a counterexample in dimension 3). Then the hypothesis of transitivity can be omitted if the dimension is higher than 3.

From Theorem 1 follows the next corollary.

Corollary 1. *Let M be a negative curvature closed surface. The time one map of the geodesic flow can not be C^1-approximated by Axiom A diffeomorphisms verifying property $\mathcal{P}$.*

In Section 1 we prove that Property $\mathcal{P}$ is a "reasonable property" and we study some properties of attractors of Axiom A diffeomorphism close to the time one map of a transitive Anosov flow. In Section 2 we introduce maps which will play an important role in the proofs of the theorems and we examine some basic facts about them. Section 3 deals with the continuity of the above mentioned maps. In Section 4 we prove that there is a repeller basic set which is a hypersurface and we complete the proofs of the Theorem in Section 5.

1 Properties of basic sets

We begin recalling some basic definitions about flows and diffeomorphisms.

Definition 1. *A compact ϕ_t-invariant set, $\Lambda \subset M$, is called a* **hyperbolic set for the flow** *ϕ if there exist a Riemannian metric on an open neighborhood $\mathcal{U}$ of Λ, and $\lambda < 1 < \mu$ such that for all $x \in \Lambda$ there is a decomposition*

$$T_x(M) = E^s_x \oplus E^u_x \oplus E^0_x$$

such that $\partial_t \phi(x,t)|_{t=0} \in E^0_x - \{0\}$, $dim(E^0(x)) = 1$, $D_x\phi_t(x)(E^i_x) \subset E^i_{\phi(x,t)}$, with $i = s, u$, and

$$\|D_x\phi(x,t)|_{E^s(x)}\| \leq \lambda^t \text{ with } t \geq 0$$

$$\|D_x\phi(x,t)|_{E^u(x)}\| \leq \mu^t \text{ with } t \leq 0.$$

A C^r flow $\phi : M \times \mathbb{R} \to M$, is called an **Anosov flow** *if M is a hyperbolic set for ϕ.*

Let $f : M \to M$ be a C^r diffeomorphism .

Definition 2. *An f-invariant set Λ is called* **hyperbolic** *if there exists a Df-invariant decomposition of $T_\Lambda M$ such that*

$$T_\Lambda M = E^s \oplus E^u$$

and $Df|E^s$ is uniformly contracting and $Df|E^u$ is uniformly expanding. More precisely, there are $c > 0$, λ, with $0 < \lambda < 1$ such that for all $x \in \Lambda$

$$\|D_x f^n|E^s(x)\| < c\lambda^n$$

and

$$\|D_x f^{-n}|E^u(x)\| < c\lambda^n.$$

A diffeomorphism $f : M \to M$ is called an **Anosov diffeomorphism** *if M is a hyperbolic set for f.*

Let $f_1 : M \to M$, the time one diffeomorphism of ϕ defined as

$$f_1(x) = \phi(x, 1),\ \forall x \in M,$$

where $\phi : M \times \mathbb{R} \to M$ is a codimension one Anosov flow if $dim(M) > 3$ (In the case that $dim(M) = 3$, codimension one property is replaced by transitivity.) Without loss of generality we may assume $dimE^s(x) = n-2$ and $dimE^u(x) = 1$ for all $x \in M$.

Since ϕ has no singularities, it follows that there exist f_1-invariant foliations $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$, $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$ and $\mathcal{F}^c$. Notice that the leaf of $\mathcal{F}^c$ through x is the same as the ϕ-orbit of x, and we denote it by $F^c(x)$ or $W^c_\phi(x)$.

By well known properties of transitive Anosov flows, we have that

$$\{F^c(x)|F^c(x) \text{ is a closed set }\} \text{ is dense in } M.$$

$$\{F^c(x)|F^c(x) \text{ is dense in } M\} \text{ is a residual set.}$$

If $\mathcal{O}$ is a periodic orbit of ϕ, then $W^s(\mathcal{O})$ consists of all points whose foward ϕ orbits never stay far from $\mathcal{O}$ and $W^u(\mathcal{O})$ of all points whose reverse ϕ orbits never stay far from $\mathcal{O}$. Both of them are dense in M, and so are $F^{cs}(x)$ and $F^{cu}(x)$ $\forall x \in \mathcal{O}$.

Since f_1 is C^r, we have that the leaves of $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$ and $\mathcal{F}^c$ are C^r. Let $f : M \to M$ be a diffeomorphism C^1-close to f_1. The map f is plaque expansive (see [7]), there exist $\mathcal{F}^{cs}_f$, $\mathcal{F}^{cu}_f$ and $\mathcal{F}^c_f$ and there is a homeomorphism $h : M \to M$ close to the identity such that if $h(x) = x'$, then $F^c_f(x')$ is C^1-close to $F^c_{f_1}(x)$ in compact sets and the manifolds $F^{cs}_f(x')$ and $F^{cs}_{f_1}(x)$ are C^1-close in compact sets. In addition,

$$hof_1(F^c_{f_1}(x)) = foh(F^c_{f_1}(x)).$$

The map f is normally hyperbolic at $\mathcal{F}^c_f$, therefore every leaf of $\mathcal{F}^c_f$ is invariant and every periodic point of f is in a closed leaf of $\mathcal{F}^c_f$.

According to what was mentioned above we have that

$$\{F^c_f(x) | F^c_f(x) \text{ is a closed set}\} \text{ is dense in } M$$

and

$$\{\mathcal{F}^c_f(x) | F^c_f(x) \text{ is dense in } M\} \text{ is a residual set.}$$

Let us denote by $F^c_f(x)$ or by $W^c(x)$ the leaf of the central foliation through the point x.

We recall that a diffeomorphism $f : M \to M$ satisfies Axiom A if the non-wandering set $\Omega(f)$ is hyperbolic and the set of periodic points is dense in $\Omega(f)$.

From now on we will assume that f is an Axiom A diffeomorphism C^1-close to f_1. Moreover, we will make the following assumption: the number of periodic points in the connected component of $W^c(x)$, between x and $f(x)$ is constant, for all f-periodic point in M. We will consider the number of periodic points in $W^c(x)$, between x and $f(x)$, in such a way that the length of this curve is almost of the same length of the trajectory $\phi(\hat{x}, t)$ of the Anosov flow, with t varying between 0 and 1, and $\hat{x}$ being a f_1 periodic point near x. Sometimes we have to consider the number of periodic points when the segment of the curve between x and $f(x)$ winds around itself more than once. The last property will be called property $\mathcal{P}$. We will prove that this property is verified in an open and dense set of the manifold.

Let $\mathcal{O} = F^c_f(x)$ where $F^c_f(x)$ is a closed curve.

The rotation number of f must be rational, because if it were irrational, there would be an hyperbolic minimal set $I \subset \mathcal{O}$ and it would be included in a basic set Λ.
If $\mathcal{O} \subset \Omega(f)$ then $\mathcal{O}$ would be in a basic set and $f|_{\mathcal{O}}$ would be expansive which leads to a contradiction with the nonexistence of one dimensional expansive diffeomorphism. Let $y \in \mathcal{O}$ then $\alpha(y) = \omega(y) = I$, hence

$$y \in W^s(I) \cap W^u(I) \subset W^s(\Lambda) \cap W^u(\Lambda),$$

therefore $y \in \Omega(f)$ which is a contradiction.

Then, there exist at least two periodic points in $\mathcal{O}$ because f is an Axiom A diffeomorphism. All the points in $\Omega(f) \cap \mathcal{O}$ must be periodic because if there were a nonperiodic point, $x \in \Omega(f) \cap \mathcal{O}$ then the invariance of $\Omega(f) \cap \mathcal{O}$ implies that $\alpha(x)$ and $\omega(x)$ would be periodic points of different indices so they would be in different basic sets.

From now on, we choose an orientation for $\mathcal{F}^c$, and denote C_b^a the curve included in a central foliation leaf, between a and b. We will consider the connected component of $\mathcal{F}^c(a)$ between a and b in the positive direction from a, in the case that $\mathcal{F}^c(a)$ is a closed curve.

Proposition 1.1. *There exists an open and dense set $V \subset M$ such that property $\mathcal{P}$ is verified for $f|V$ i.e. all periodic points in V have the same number of periodic points in the connected component of $W^c(x)$, between x and $f(x)$.*

Proof. The metric induced by the Riemannian metric on the leaves of $\mathcal{F}^c$ will be denoted d^c.

The lengths of the curves $C_{f(x)}^x$ are bounded away from 0, and as f is Axiom A there exists κ such that $d^c(p, q) > \kappa$, if p and q are periodic points in the same leaf of $\mathcal{F}^c$. Then, there exists $m \in \mathbb{N}$ such that

$$m = \min\{n \in \mathbb{N} : W^c(x) \text{ has exactly } n \text{ periodic points in } C_{f(x)}^x\}.$$

Let p a periodic point verifying that the number of periodic points in $C_{f(p)}^p$ is m.

We claim that there is an open neighborhood U of $C_{f(p)}^p$ such that for all periodic point x in U the number of periodic points in $C_{f(x)}^x$ is m.

If not, there exists a sequence of periodic points $p_n \to p$ such that the number of periodic points in $C_{f(p_n)}^{p_n}$ is greater than m, so there exist more than m limit points in $C_{f(p)}^p$. Since these limit points must be periodic, this contradicts our assumption.

Therefore, there exists a curve included in a dense leaf of central foliation in U. So, if we saturate U by the central foliation we have an open and dense set such that any periodic point q in it has exactly m periodic points in $C_{f(q)}^q$. □

Let us recall that there exists a finite number of attractors (repellers) whose basin of attraction (repulsion) are open since f is Axiom A.

Here are some elementary properties of attractor basic sets.

Let $\mathcal{A}$ denote an attractor basic set of the spectral decomposition of f. Notice that $\mathcal{A} \neq M$ because f can not be an Anosov diffeomorphism. There is no loss of generality if we consider that $\mathcal{A}$ is connected.

Lemma 1.1. $Dim(W^s(x)) = n - 1, \forall x \in \mathcal{A}$.

Proof. We have assumed that $dim(E^s_\phi) = n - 2$, then as f is C^1-close to f_1 we have that $dim(W^s(x)) = n - 1$ or $dim(W^s(x)) = n - 2$ for all $x \in \Omega(f)$.

Let $x \in \mathcal{A} \cap per(f)$, where $per(f)$ is the set of f-periodic points.

Suppose that $dim(W^s(x)) = n - 2$.

Since $\mathcal{A}$ is an attractor, $W^u(x) \subset \mathcal{A}$; hence $F^c_{loc}(x) \subset W^u(x) \subset \mathcal{A}$. The set $\mathcal{A}$ is closed and f-invariant so there exists $x' \in F^c(x) \cap \mathcal{A} \cap per(f)$.

But $dim(W^s(x')) = n - 1$ since $dim(W^s(x)) = n - 2$. It follows that there exist two periodic points of different indices in $\mathcal{A}$, which is impossible. □

Lemma 1.2. *For every closed curve $\mathcal{O}$ in $\mathcal{F}^c$ there exists a periodic point $p \in \mathcal{A} \cap \mathcal{O}$.*

Proof. Since $\mathcal{O}$ is closed, $W^s(\mathcal{O})$ is dense in M and $W^s(\mathcal{A})$ is an open set, there exist y in $W^s(\mathcal{O}) \cap W^s(\mathcal{A})$ and $y' \in W^{ss}(y) \cap \mathcal{O}$ such that $y' \in W^s(\mathcal{A})$.

As $y' \in \mathcal{O}$, $y' \in W^s(p)$ for a periodic point $p \in \mathcal{O}$. Then $p \in \mathcal{A} \cap \mathcal{O}$. □

Let $K = \max_{x \in M} length(C^x_{f(x)})$. K is finite because M is compact and the map $g : M \to \mathbb{R}$ such that every $x \in M$ is mapped into the length of $C^x_{f(x)}$ is continuous.

The previous lemma asserts that in every segment γ of central closed curve with $length(\gamma) \geq K$, there exists a periodic point $p \in \gamma \cap \mathcal{A}$.

Corollary 1.1. *Every leaf of $\mathcal{F}^c$ intersects $\mathcal{A}$.*

Proof. Let $\gamma \subset \mathcal{F}^c$ with $length(\gamma) \geq K$. Since

$$\{F^c_f(x) | F^c_f(x) \text{ is a closed set}\} \text{ is dense in } M,$$

we can choose arcs γ_n such that γ_n are included in closed leaves of $\mathcal{F}^c$, $\gamma_n \to \gamma$, and $length(\gamma_n) \geq K$. Then, there exists a sequence (p_n) such that $p_n \in \mathcal{A} \cap \gamma_n$, and any of its limit points $p \in \gamma \cap \mathcal{A}$. □

Lemma 1.3. *In every leaf of $\mathcal{F}^c_f$ there exists at least one point outside of $W^s(\mathcal{A})$.*

Proof. If $F_f^c(x)$ is closed, by Lemma 1.2 we have that there exists a periodic point $p \in \mathcal{A} \cap F_f^c(x)$ and by Lemma 1.1 $dim(W^s(p)) = n-1$. The hyperbolicity of f implies that there exists a periodic point $q \in F_f^c(x)$ such that $dim(W^s(q)) = n-2$, hence $q \in \Sigma$ where Σ is a basic set of f, $\Sigma \neq \mathcal{A}$. So we proved the claim in the case that $F_f^c(x)$ is closed.

In the case that $C_0 = F_f^c(x)$ is a future-dense curve, this is, if $f(x) > x$ in the chosen orientation, then $W^{c+}(x) = \{y \in W^c(x)/y \geq x\}$ is dense, and if $f(x) < x$ then $W^{c-}(x)$, with the obvious definition, is dense.

Clearly we have that $C_0 \cap W^s(\mathcal{A}) \neq \emptyset$.

We only need to show that C_0 is not included in $W^s(\mathcal{A})$, i.e. $C_0 \cap \partial(W^s(\mathcal{A})) \neq \emptyset$.

Suppose that for every y in $C^x_{f(x)}$, we have that $y \in W^s(\mathcal{A})$. There exists an open and nondense set U, such that $\mathcal{A} \subset U$, $f(U) \subset U$ and $C^x_{f(x)} \subset U$; then if C_0 intersects U, C_0 would be included in U in the future. This contradicts the nondensity of U, so there exists $y \in C^x_{f(x)}$ such that $y \notin W^s(\mathcal{A})$.

It still remains to prove the claim in the case that $C = F_f^c(x)$ is any curve.

Recall that $K = \max_{x \in M} length(C^x_{f(x)})$.

Suppose that there exists a curve $\gamma \subset F_f^c(x)$ such that $\gamma \subset W^s(\mathcal{A})$ and $length(\gamma) \geq K + 1$.

Then there exists an open set V, $V \subset W^s(\mathcal{A})$ and $\gamma \subset V$. There exists $y \in V$ such that $W^c(y)$ is dense in M, and $W^c(y) \cap V$ has length greater or equal than K. This gives the existence of a fundamental domain in $W^c(y) \cap V$, and then in $W^s(\mathcal{A})$. This contradicts the previous case. □

Note that we have proved that every leaf of the central foliation "goes away" from the basin of attraction of any attractor.

Lemma 1.4. *No curve γ, γ included in $F_f^c(x)$ for any x, satisfies $\gamma \subset \mathcal{A}$.*

Proof. Suppose the statement is false, i.e. there exists $\gamma \subset W^c_{loc}(x)$ such that $\gamma \subset \mathcal{A}$. Since $\gamma \subset \mathcal{A} \subset W^s(\mathcal{A})$, then the negative iterates of γ are included in $\mathcal{A}$ and the length of them grow exponentially.

Let $z \in \alpha(x)$ then $z \in \mathcal{A}$ and by the proof of Lemma 1.3 $W^c(z)$ has to intersect $\partial(W^s(\mathcal{A}))$, but $W^c(z) \subset \mathcal{A} \subset W^s(\mathcal{A})$, which yields a contradiction. □

All the above lemmas admit versions for repeller basic sets and the proofs are analogous. In fact, if Λ is a repeller basic set, then for $x \in \Lambda$, $Dim(W^s(x)) = n-2$, every leaf of $\mathcal{F}_f^c$ intersects Λ, in every leaf of $\mathcal{F}_f^c$ there exists a point outside of $W^u(\Lambda)$, and no γ included in $F_f^c(x)$ satisfies $\gamma \subset \Lambda$.

2 Properties of the projection along the central foliation

In this section, we will introduce some maps which are important from the technical point of view.

Definition 2.1. Let $S_A : W^s(\mathcal{A}) \to \partial W^s(\mathcal{A})$ be a map such that, for every x in the basin of the attractor $\mathcal{A}$, $S_A(x)$ is the nearest point in its central leaf in the positive direction verifying that it is not in the basin of attraction of $\mathcal{A}$.

Definition 2.2. Let $\tilde{S}_A : W^s(\mathcal{A}) \to \partial W^s(\mathcal{A})$ be the map analogous to S_A, but in the negative direction of the central foliation .

Definition 2.3. Let $S : \mathcal{A} \to \partial W^s(\mathcal{A})$ be the restriction of S_A to $\mathcal{A}$ and $\tilde{S} : \mathcal{A} \to \partial W^s(\mathcal{A})$ the restriction of $\tilde{S}_A$ to $\mathcal{A}$.

Lemma (1.3) makes the preceding definitions possible.

Let $\widetilde{W^c(x)}$ denote the connected component of $W^c(x) \cap W^s(\mathcal{A})$ which contains x.

Let $l : \mathcal{A} \to \mathbb{R}$, $l(x) = length(C^x_{S(x)})$.

Lemma 2.1. *l is lower semicontinuous.*

Proof. Since $C^x_{S(x)} - \{S(x)\} \subset W^s(\mathcal{A})$ and $W^s(\mathcal{A})$ is an open set, there exists a neighborhood V such that $C^x_{S(x)} - \{S(x)\} \subset V \subset W^s(\mathcal{A})$.
The central foliation is a C^1- lamination because f is C^1-close to the time one map of an Anosov flow (see [7]), hence for all $\epsilon > 0$ there exists a neighborhood U_x of x such that if $y \in U_x$ then the curve $C^y_{y'}$ included in $\mathcal{F}^c(y)$ with $length(C^y_{y'}) = l(x) - \epsilon$ is included in V, and hence in $W^s(\mathcal{A})$. Then $l(y) \geq l(x) - \epsilon$ which proves that l is a semicontinuous map. □

Since $l : \mathcal{A} \to \mathbb{R}$ is semicontinuous, the set R of points of continuity of l is a residual set. Let $\Phi : M \times \mathbb{R}_{\geq 0} \to M$ such that $\Phi(x, l) = z$, if $z \in W^c(x)$, z is in the positive direction of $W^c(x)$ and $length(C^x_z) = l$. Φ is a continuous map then

$$S(x) = \Phi(x, l(x))$$

is continuous over R.

Without loss of generality we can assume that R is a residual set of continuity for both S and $\tilde{S}$.

Analogously there exists a residual set Q in $W^s(\mathcal{A})$ such that Q is a set of continuity for S_A and $\tilde{S}_A$.

Following, we prove some properties of the map S. They are verified by $\tilde{S}$ and the proofs are analogous.

Lemma 2.2. *$S(R)$ is f-invariant.*

Proof. Let $x \in R$, $y = S(x)$. For all $z \in C_y^x - \{y\}$, we have that $z \in W^s(\mathcal{A})$, $f(z) \in W^c(f(x))$ and $f(z) \in W^s(\mathcal{A})$. From $f(y) \in \partial W^s(\mathcal{A})$ it follows that $f(y) = S(f(x))$. Replacing f by f^{-1} we conclude that

$$f(S(R)) = S(R). \qquad \square$$

Lemma 2.3. *For all $y \in S(R)$, $dim(W^s(y)) = n - 2$.*

Proof. Let $y = S(x)$ with $x \in \mathcal{A}$; since $dim(W^{ss}(y)) = n - 2$ and $dim(W^{uu}(y)) = 1$, $dim(W^s(y)) = n - 1$ or $n - 2$, but by Lemma (1.1) if $z \in C_y^x - \{y\}$ then $z \in W^s(x)$. Then

$$W_\epsilon^c(y) = \{z \in W^c(y) \text{ such that } d^c(z, y) < \epsilon\}$$

can not be included in $W^s(y)$ and we can assert that $dim(W^s(y)) = n - 2$. $\square$

Lemma 2.4. *The set of periodic points in $\mathcal{A} \setminus R$ is nowhere dense in $\mathcal{A}$.*

Proof. In order to prove the lemma it is enough to prove:

Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of periodic points such that S is not continuous at p_n and $p_n \to x$. Then S is not continuous at x.

Let $q_n = S(p_n)$.

Since p_n is a point of discontinuity, there exist $\alpha > 0$ and $(r_{n_k}) \subset \mathcal{A}$ such that $\lim_{k\to\infty} r_{n_k} = p_n$ and

$$length(C_{S(r_{n_k})}^{r_{n_k}}) > length(C_{S(p_n)}^{p_n}) + \alpha$$

and for any ϵ with $0 < \epsilon < \frac{\alpha}{2}$ there exist $(s_{n_k}) \subset R$ such that $\lim_{k\to\infty} s_{n_k} = p_n$ and

$$length(C_{S(s_{n_k})}^{s_{n_k}}) \geq length(C_{S(r_{n_k})}^{r_{n_k}}) - \epsilon > length(C_{S(p_n)}^{p_n}).$$

It follows that there exists a periodic limit point of $S(s_{n_k})$, q'_n, in $W^c(p_n)$.
Both q_n and q'_n are in $W^c(p_n) \cap \overline{S(R)}$, are periodic and

$$dim(W^s(q_n)) = dim(W^s(q'_n)) = n - 2.$$

Since q_n and q'_n are in the same closed leaf of $\mathcal{F}^c$, it follows that there exists a periodic point p'_n, such that $p'_n \in C^{q_n}_{q'_n}$ and $dim(W^s(p'_n)) = n - 1$.

Suppose, contrary to our claim, that S is continuous at x.
From $p_n \to x$ we conclude that $q_n \to S(x)$ by the continuity of S at x.
Besides $q'_n \to S(x)$ because there exist $(s_{n_k}) \subset R$ such that $\lim_{k\to\infty} s_{n_k} = p_n$ and $\lim_{k\to\infty} S(s_{n_k}) = q'_n$. Letting a convenient subsequence $k(n)$, we can assert that

$$\lim_{n\to\infty} s_{n_{k(n)}} = x \text{ and } \lim_{n\to\infty} S(s_{n_{k(n)}}) = S(x)$$

by the continuity of S at x. This gives $q'_n \to S(x)$.

Then $dist(q_n, q'_n) \to 0$ when $n \to \infty$ and $d^c(q_n, q'_n) \to 0$ when $n \to \infty$.
But $d^c(q_n, q'_n) > \min\{d^c(p'_n, q'_n), d^c(p_n, q'_n)\}$ and this leads to a contradiction because p'_n and q'_n (or p_n and q'_n) are in different basic sets because they have different indices.
We have proved that S is not continuous at x.

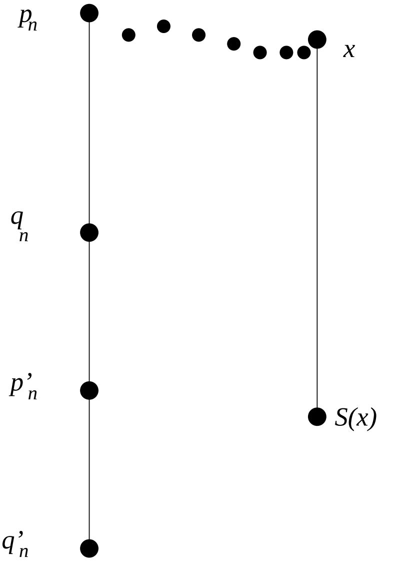

Figure 1

□

Observe that as a consequence we have that for all $x \in \mathcal{A}$ there exists a sequence of periodic points $(p_n)_{n\in\mathbb{N}} \subset R$ such that $p_n \to x$.

Lemma 2.5. *$\overline{S(R)}$ is transitive and $\overline{S(R)} \subseteq \Omega(f)$.*

Proof. Since $\mathcal{F}^c$ is continuous, the set of periodic points is dense in R and $S(p)$ is periodic if p is periodic, then the set of periodic points is dense in $S(R)$, hence

$$S(R) \subseteq \Omega(f).$$

Analogously the image of a dense orbit is dense in $S(R)$. □

Corollary 2.1. *From the above properties we conclude that $\overline{S(R)}$ is included in Λ, a basic set of the spectral decomposition of f.*

Lemma 2.6. $S(W^s(x)) \subset W^s(S(x))$.

Proof. Let $x \in \mathcal{A}$, $y \in W^s(x) \cap \mathcal{A}$. Suppose that $S(y) \notin W^s(S(x))$. Since $S(y) \in F^{cs}(x)$ there exists $z = W^s(S(y)) \cap W^c(x)$. We have that $\forall w \in \partial(W^s(\mathcal{A}))$, $W^s(w) \subset \partial(W^s(\mathcal{A}))$, then $W^s(S(x)) \subset \partial(W^s(\mathcal{A}))\ \forall x \in \mathcal{A}$, and $z \in \partial(W^s(\mathcal{A}))$, but this contradicts the definition of S. □

Lemma 2.7. *If x is a point of continuity of S, then all the points in $W^s(x) \cap \mathcal{A}$ are continuity points of S.*

Proof. Let x be a point of continuity of S, $y \in W^s_{loc}(x) \cap \mathcal{A}$. We first prove that y is a continuity point of S.
Let $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, such that $\lim_{n\to\infty} y_n = y$. There exists $x_n = W^s_{loc}(y_n) \cap W^u(x)$ and $y_n \in W^s(x_n)$. By continuity of the stable foliation, we have $\lim_{n\to\infty} x_n = x$, and by continuity of S at x we conclude that $\lim_{n\to\infty} S(x_n) = S(x)$.
From $y_n \in W^s(x_n)$, and the above lemma, it follows that $S(y_n) \in W^s(S(x_n))$, hence $S(y_n) = W^s_{loc}(S(x_n)) \cap W^c(y_n)$.
By the continuity of W^s and W^c we have that:

$$\lim_{n\to\infty} W^s_{loc} S(x_n) = W^s_{loc} S(x) \text{ and } \lim_{n\to\infty} W^c(y_n) = W^c(y);$$

hence

$$\lim_{n\to\infty} S(y_n) = W^s_{loc} S(x) \cap W^c(y) = S(y).$$

We have proved that $\forall y \in W^s_{loc}(x) \cap \mathcal{A}$, S is continuous at y i.e. $S|_{W^s_{loc}(y)\cap\mathcal{A}}$ is continuous.
Now, if $z \in W^s(x) \cap \mathcal{A}$ there is $N > 0$ such that $f^N(z) \in W^s_{loc}(f^N(x)) \cap \mathcal{A}$ and the previous argument still applies. □

Remark. Note that Lemmas (2.6) and (2.7) are verified not only by S and $\tilde{S}$ but also by S_A and $\tilde{S}_A$. The proofs are analogous.

Lemma 2.8. *If $x \in \mathcal{A}$, then x is a point of continuity of S if and only if x is a point of continuity of S_A.*

Proof. We only have to prove that if $x \in \mathcal{A}$ is a point of continuity of S then it is a continuity point of S_A.
Let y be a point close to x, then $y' = W^u_{loc}(x) \cap W^s_{loc}(y)$ is a point in $\mathcal{A}$ such that $S(y')$ is close to $S(x)$ and

$$S_A(y) = W^s(S(y')) \cap W^c(y) \text{ is close to } S_A(y') = S(y').$$

Hence $S_A(y)$ is close to $S_A(x) = S(x)$. □

Proposition 2.1. *If f satisfies property $\mathcal{P}$ then for every periodic point p, S is continuous at p.*

Proof. Let k denote the number of periodic points in $C^x_{f(x)}$, for all periodic point $x \in \mathcal{A}$. Suppose x is a periodic discontinuity point of S, then we have a sequence $(x_n)_{n\in\mathbb{N}}$ of periodic points of continuity such that $\lim_{n\to\infty} x_n = x$ and $length(C^{x_n}_{S(x_n)}) > length(C^x_{S(x)}) + \alpha$, with $\alpha > 0$.
For every x_n, there exist k periodic points $x^1_n < \ldots < x^k_n$ in $C^{x_n}_{f(x_n)}$, ordered by the chosen orientation.
Since $\lim_{n\to\infty} W^c(x_n) = W^c(x)$ in compact sets, there exist x^i, limit point of x^i_n in $W^c(x)$, and x^i must be periodic. Since the number of periodic points in $C^{x_n}_{f(x_n)}$ and in $C^x_{f(x)}$ is the same, then there exists only a limit point of x^i_n, i.e. $\lim_{n\to\infty} x^i_n = x^i$.
In particular $\lim_{n\to\infty} x^1_n = x^1$, and this gives $\lim_{n\to\infty} S(x_n) = S(x)$; so $length(C^{x_n}_{S(x_n)}) < length(C^x_{S(x)}) + \alpha$ if n is big enough, which is absurd.
We have proved that S is continuous at every periodic point. □

Lemma 2.9. *Let Λ be a basic set and $x \in \Lambda$.*

1. *If $dim(W^s(x)) = n-1$ then there is a finite number of points of Λ in the connected component of $W^c(x) \cap W^s(\Lambda)$ that contains x.*
2. *If $dim(W^s(x)) = n-2$ then there is a finite number of points of Λ in the connected component of $W^c(x) \cap W^u(\Lambda)$ that contains x.*

Proof. We will prove just the first statement.
Suppose that it is false. Then we can choose $\{x_i\}$ in $\Lambda \cap W^s(\Lambda) \cap W^c(x)$, such that $x_1 < x_2 < \ldots < x_l < \ldots$ in the given orientation of $W^c(x)$. There exists $k > 0$ such that $f^{-1}|_{W^c_k(x)}$ "expands", $\forall x \in \mathcal{A}$. Then there exists $n_1 \in \mathbb{N}$ verifying that $length(f^{-n_1}(C^x_{x_1})) > k$, for all $n \geq n_1$. There exists $n_2 \in \mathbb{N}$ such that $length(f^{-n_2}(C^{x_1}_{x_2})) > k$, for all $n \geq n_2$. Let l_0 such that $kl_0 > K+1$, where $K = \max_{x \in M} length(C^x_{f(x)})$. We continue in this way obtaining $n_3, \ldots, n_{l_0}$. Let $N = max\{n_1, \ldots, n_{l_0}\}$, then

$$length(f^{-N}(C^x_{x_{l_0}})) > kl_0 > K+1$$

Hence, as in the proof of Lemma 1.3 we conclude that there exists $p \in f^{-N}(C^x_{x_{l_0}})$ such that $p \in \partial W^s(\Lambda)$ and therefore $f^N(p) \in \partial W^s(\Lambda)$ and $f^N(p) \in C^x_{x_{l_0}} \subseteq W^s(\Lambda)$; which is a contradiction.
We have actually proved that there are no more than $[\frac{K+1}{k}]$ points of Λ in the connected component of $W^s(\Lambda) \cap W^c(x)$. □

3 Continuity of the map *S*.

Let us first prove the next lemma.

Lemma 3.1. *Let x be a continuity point of S_A and $\tilde{S}_A$, (i.e. $x \in Q$) then for all $y \in W^s(x)$, $\widetilde{W^c(y)} \cap \mathcal{A} \neq \emptyset$.*

Proof. Let $\epsilon > 0$ be such that $\cup_{x \in \mathcal{A}} W^s_\epsilon(x) \subset W^s(\mathcal{A})$.
Let $x \in Q$ and U_x be a neighborhood of x such that for all $y \in U_x$ we have that $length(\widetilde{W^c(y)})$ is close enough to $length(\widetilde{W^c(x)})$, and let $y \in U_x \cap W^s(x)$. Since $\widetilde{W^c(y)} \subset W^s(\mathcal{A})$ and $W^s(\mathcal{A})$ is open, there exists a neighborhood of $\widetilde{W^c(y)}$, $\mathcal{V}$, such that $\mathcal{V} \subseteq W^s(\mathcal{A})$ and $\mathcal{V} \subset \cup_{z \in \mathcal{U}_x}(\widetilde{W^c(z)})$, in such a way that if $z \in \mathcal{V} \cap \mathcal{A}$ then $length(\widetilde{W^c(z)})$ is close enough to $length(\widetilde{W^c(y)})$.

By the density of the closed leaves in the central foliation, there exists a curve ζ in $\mathcal{V}$, included in a closed leaf of the central foliation, $\mathcal{O}$ such that $\zeta = \mathcal{O} \cap W^s(\mathcal{A})$.

There exists a periodic point p such that $p \in \zeta \cap \mathcal{A}$, $\zeta = \widetilde{W^c(p)}$ and since S_A and $\tilde{S}_A$ are continuous at y by the remark of Lemma 2.7, the lengths of $\widetilde{W^c(y)}$ and ζ are close; and the lengths of the curves $C^p_{S_A(p)}$, and $C_p^{\tilde{S}_A(p)}$ are greater than the ϵ previously defined.

Then, considering open sets $\mathcal{V}_n$ such that $\mathcal{V}_n \to \widetilde{W^c(y)}$, we can assert that there exist curves $\zeta_n \subset \mathcal{V}_n$ and periodic points $p_n \in \zeta_n \cap \mathcal{A}$ such that the lengths of $\widetilde{W^c(y)}$ and ζ_n are close; and the lengths of the curves $C^{p_n}_{S_A(p_n)}$, and $C_{p_n}^{\tilde{S}_A(p_n)}$ are greater than ϵ.

Since ζ_n converges to $\widetilde{W^c(y)}$ and the distance of p_n to $\partial(W^s(\mathcal{A}))$ is bounded away from 0, there exists a limit point p of p_n such that $p \in \mathcal{A} \cap \widetilde{W^c(y)}$.

We have proved that if $x \in Q$ then

$$\forall y \in W^s_{loc}(x), \exists p \in \widetilde{W^c(y)} \cap \mathcal{A}.$$

Successive applications of this proceeding enables us to conclude that if $x \in Q$

$$\forall y \in W^s(x), \exists p \in \widetilde{W^c(y)} \cap \mathcal{A}. \qquad \square$$

Corollary 3.1. $\Lambda = \overline{S(R)}$ *is a repeller set.*

Proof. Let $x \in Q \cap \mathcal{A}$, $z \in W^s(S(x))$ and $z' = W^c(z) \cap W^{ss}(x)$. Since $z' \in W^s(x)$ with $x \in Q$, then by Lemma 3.1 there exists $q \in \widetilde{W^c(z')} \cap \mathcal{A}$; hence $S(q) = z$ and $z \in S(R)$. Then

$$\forall x \in Q \cap \mathcal{A}, W^s(S(x)) \subseteq S(R).$$

We have proved that $\overline{S(R)}$ is included in a basic set Λ. Now, if $y = S(x)$ with $x \in \mathcal{A} \cap Q$ then

$$W^s(y) \subseteq S(R) \subseteq \overline{S(R)} \subseteq \Lambda \subseteq \overline{W^s(y)}.$$

It follows that $\overline{S(R)}$ is a basic set, and since it contains a stable manifold we have that $\Lambda = \overline{S(R)}$ is a repeller set. $\square$

Let us consider the following maps.

Definition 3.1. *Let* $\Sigma_\Lambda : W^u(\Lambda) \to \partial W^u(\Lambda)$ *be a map such that, for every* x *in the basin of repulsion of* Λ*,* $\Sigma_\Lambda(x)$ *is the nearest point in its central leaf in the positive direction verifying that it is not in the basin of repulsion of* Λ*.*

Definition 3.2. *Let* $\tilde{\Sigma}_\Lambda : W^u(\Lambda) \to \partial W^u(\Lambda)$ *be the map analogous to* Σ_Λ*, but in the negative direction of the central foliation.*

Definition 3.3. *Let* $\Sigma : \Lambda \to \partial W^u(\Lambda)$ *be the restriction of* Σ_Λ *to* Λ *and* $\tilde{\Sigma} : \Lambda \to \partial W^u(\Lambda)$ *the restriction of* $\tilde{\Sigma}_\Lambda$ *to* Λ*.*

The version of Lemma (1.3) for repeller sets makes the preceding definitions possible.

As done after Definition 2.3 we define $\widetilde{W^c(x)}$ as the connected component of $W^c(x) \cap W^u(\Lambda)$ which contains x, if $x \in W^u(\Lambda)$.

All the properties verified by S, $\tilde{S}$, S_A and $\tilde{S}_A$ are verified by Σ, $\tilde{\Sigma}$, Σ_Λ and $\tilde{\Sigma}_\Lambda$ with the obvious modifications. In particular, there exists a residual set $\Theta \subset W^u(\Lambda)$ such that Σ_Λ and $\tilde{\Sigma}_\Lambda$ are continuous in Θ. Besides, if $x \in \Theta$ then for all $y \in W^u(x)$ we have that $\widetilde{W^c(y)} \cap \Lambda \neq \emptyset$. Once again, if property $\mathcal{P}$ is verified, all the periodic points of Λ are continuity points for all these maps.

Lemma 3.2. *Let* $x \in \Lambda$*. Suppose that* $y \in W^u(x)$*. Then*

$$\widetilde{W^c(y)} \cap \Lambda \neq \emptyset.$$

Proof. By the version of Lemma 3.1 for repeller sets and the continuity of Σ_Λ and $\tilde{\Sigma}_\Lambda$ restricted to Θ, we have that for all point $x \in \Theta$ there is a neighborhood U_x such that if $y \in U_x$ and $z \in W^u_{loc}(y)$ then $\widetilde{W^c(z)} \cap \Lambda \neq \emptyset$.
Let

$$\mathcal{U} = \cup_{x \in \Theta} U_x.$$

$\mathcal{U}$ is an open and dense set in $W^u(\Lambda)$.
Let $x \in \Lambda$ and suppose by contradiction that there exists $y_0 \in W^u(x)$ such that $\widetilde{W^c(y_0)} \cap \Lambda = \emptyset$. In addition, there exists a neighborhood V_{y_0} of y_0 such that if $z \in V_{y_0} \cap W^u(y_0)$ then $\widetilde{W^c(z)} \cap \Lambda = \emptyset$.
Since $W^u(x)$ is dense in Λ, there exists $v \in W^u(x) \cap \mathcal{U}$, hence there exists $\tilde{v} \in W^{uu}(x) \cap \mathcal{U}$.
Let $\mathbf{C} \subseteq W^{uu}(x)$ an arc such that is maximal with respect to the following property: if $y \in \mathbf{C}$, $\widetilde{W^c(y)} \cap \Lambda = \emptyset$.

Let $\tilde{r}$ be an extreme of $\mathbf{C}$ and $r = \widetilde{W^c(\tilde{r})} \cap \Lambda$.
If $w \in \mathbf{C} \cap W^{uu}_{loc}(\tilde{r})$ then $\overline{w} = W^c(w) \cap W^{uu}_{loc}(r)$ exists and verifies that $\widetilde{W^c(\overline{w})} \cap \Lambda = \emptyset$; so we can define

$$W^{u+}(r) = \text{ connected component of } \{y \in W^u(r) | \widetilde{W^c(y)} \cap \Lambda = \emptyset\}$$

such that $W^{u+}(r) \cap W^u_\epsilon(r) \neq \emptyset$ for any $\epsilon > 0$.

For all $n \in \mathbb{N}$, $f^n(r) \in \Lambda$ and $W^{u+}(f^n(r))$ contains an arc $D_n \subset W^{uu}(f^n(r))$ whose length grows exponentially and it has an extreme in $f^n(r)$.
Let $q \in \omega(r)$, then $W^{u+}(q)$ contains a "half plane" of $W^u(q)$, i.e. with an adequate orientation $\succ$ on $W^{uu}(q)$, we have

$$W^{u+}(q) = \{ \quad v \quad \in W^u(q) |\ W^c(v) \cap W^{uu}(q) \succ q\}$$

We may also assume that $f^n(r) \to q$. Taking n and m big enough we obtain that $f^n(r)$ and $f^m(r)$ are as close as we wish, then there is no possibility that $W^s(f^n(r))$ intersects $W^u(f^m(r))$ in $W^{u+}(f^m(r))$ because this point would be in $W^{u+}(f^m(r)) \cap \Lambda$.
In the same way there is no possibility that $W^s(f^m(r))$ intersects $W^u(f^n(r))$ in $W^{u+}(f^n(r))$.
It follows that if n and m are big enough then $W^s(f^n(r))$ intersects $W^u(f^m(r))$ in $W^c(f^m(r))$ because the central-stable foliation locally separates M.
Then there are two possibilities:

1. There exist infinite many stable manifolds of $f^j(r)$, with $j \in \mathbb{N}$. In this case, there exist infinite many points in $\Lambda \cap W^c(f^m(r))$, but this contradicts Lemma 2.9.

2. There exists a finite number of different stable manifolds of $f^j(r)$, with $j \in \mathbb{N}$.
   We can suppose that $W^s(f^n(r))$ is the same for all $n \in \mathbb{N}$. Since $f^n(r) \to q$, we have that q is periodic point; and since $W^{u+}(q) \cap \Lambda = \emptyset$, q is not a continuity point of Σ_Λ and $\tilde{\Sigma}_\Lambda$, because it would contradict the version of Lemma 3.1 for repeller sets.
   On the other hand, the version of proposition 2.1 for repeller sets asserts that all periodic points in Λ are continuity points of Σ, and $\tilde{\Sigma}$, and hence of Σ_Λ, and $\tilde{\Sigma}_\Lambda$, which yields a contradiction.

   We notice that it is at this point where Property $\mathcal{P}$ is used.

We have proved that for all $x \in \Lambda$, and for all $y \in W^u(x)$

$$\widetilde{W^c(y)} \cap \Lambda \neq \emptyset.$$

□

Proposition 3.1. *$S, \tilde{S} : R \to \partial W^s(\mathcal{A})$ can be extended continuously to $\mathcal{A}$.*

Proof. We will just prove the proposition for S.
We recall that there exists a residual set R such that $S : R \to \partial W^s(\mathcal{A})$ is continuous.
If for all $y \in \mathcal{A} \setminus R$, and for all sequence $(x_n)_{n\in\mathbb{N}} \subset R$ with $\lim_{n\to\infty} x_n = y$, we have that there exists $\lim_{n\to\infty} S(x_n)$ and it is unique, then we can extend continuously S, in such a way that $S(y) = \lim_{n\to\infty} S(x_n)$.

We will show that if $(x_n)_{n\in\mathbb{N}} \subset R$ with $\lim_{n\to\infty} x_n = y$, and $(w_n)_{n\in\mathbb{N}} \subset R$ with $\lim_{n\to\infty} w_n = y$, then every subsequence verifies that

$$\lim_{i\to\infty} S(x_{n_i}) = \lim_{j\to\infty} S(w_{n_j}).$$

Since $W^c(x_n) \to W^c(y)$ in compact sets, and the lengths of the curves $C^{x_n}_{S(x_n)}$ are bounded, there exists $y' = \lim S(x_{n_i})$, $y' \in \mathcal{F}^c(y)$, $y' \in \Lambda$. Identical argument shows that there exists y'' such that $y'' = \lim S(w_{n_j})$, $y'' \in \mathcal{F}^c(y)$, and $y'' \in \Lambda$. We suppose that $y' \neq y''$ and there is no point in $C^{y'}_{y''} \cap \Lambda$ but the extremes of $C^{y'}_{y''}$, because in the connected component of $W^c(y') \cap \Lambda$ there is only a finite number of points by Lemma 2.9.

In order to prove the proposition we need the next lemma:

Lemma 3.3. *There exist $s \in \mathcal{A}$, $r, r' \in \Lambda$, and q such that $q \in \Delta$, where Δ is a basic set $\Delta \neq \Lambda$; all these points are in the same leaf of $\mathcal{F}^c$; $r \in C^s_q$, and $q \in C^r_{r'}$.*

Proof. Let $s \in \omega(y)$. There exist $(m_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \lim_{i\to\infty} f^{m_k}(x_{n_i}) = s \quad \text{and} \quad \lim_{k\to\infty} \lim_{j\to\infty} f^{m_k}(w_{n_j}) = s.$$

Since the central foliation is continuous (in compact sets) and the length of the curves $C^{f^{m_k}(x_{n_i})}_{S(f^{m_k}(x_{n_i}))}$ and $C^{f^{m_k}(w_{n_j})}_{S(f^{m_k}(w_{n_j}))}$ are bounded, there exist

$$\lim_{k\to\infty} f^{m_k}(y') = r \quad \text{and} \quad \lim_{k\to\infty} f^{m_k}(y'') = r',$$

with $r, r' \in \Lambda$, $r, r' \in \mathcal{F}^c(s)$ and $r \in C^s_{r'}$.
Since $length(C^z_{S(z)}) \leq K + 1$ for all $z \in \mathcal{A}$ (see the proof of Lemma 1.3) and $S(f(z)) = f(S(z))$ we have that $f^m(C^z_{S(z)}) = C^{f^m(z)}_{f^m(S(z))}$ for all $m \in \mathbb{N}$, and therefore $f^{m_k}(C^{y'}_{y''}) \to C^r_{r'}$.

Since $C^{y'}_{y''}$ is not included in Λ by the version of Lemma 1.4 for repeller sets, there exists $z \in C^{y'}_{y''}$ such that $z \notin \Lambda$. There exists $u \in \Omega(f)$ such that $z \in W^s(u)$, with $u \in \Delta$, where Δ is a basic set $\Delta \neq \Lambda$. It follows that $\omega(z) = \omega(u)$, hence $\omega(z) \subseteq \Delta$.
Since $f^{m_k}(C^{y'}_{y''}) \to C^r_{r'}$, there exists a point $q \in C^r_{r'} \cap \omega(z)$, therefore $q \in C^r_{r'} \cap \Delta$ and the lemma is proved. □

Let us continue with the proof of Proposition 3.1.

Let s, r, q and r' be as in the Lemma 3.3.

Since $s \in \mathcal{A}$, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ such that $z_n \in \mathcal{A}$, z_n is a continuity point of S and $\tilde{S}$, $z_n \to s$ and $S(z_n) \to r'$.
Let n_0 be big enough in order to have

$$\alpha = W^u(s) \cap W^s(z_{n_0}), \text{ and } \beta = W^u(q) \cap W^s(z_{n_0})$$

close to s and q respectively. It follows that α and β are in the same leaf of the central foliation. Let

$$\rho = W^u(r) \cap W^c(\alpha).$$

Since $W^u(r)$ has dimension 2, there exists a curve **C**, such that **C** is the connected component of $W^c(\rho) \cap W^u(\Lambda)$.

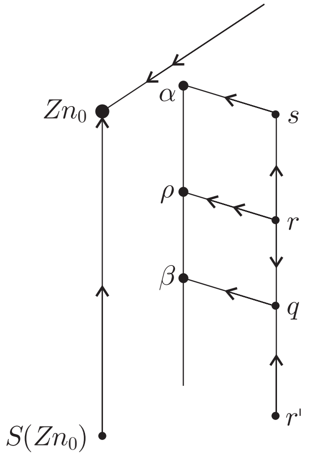

Figure 2

Since $\alpha \in \mathcal{A}$ and β is such that $\beta \in W^u(\Delta)$ where Δ is a basic set such that $q \in \Delta$ with $\Delta \neq \Lambda$, α and β are not in **C**. Hence $\mathbf{C} \subset C^{\alpha}_{\beta}$.
From Lemma 3.2 we have that there exists $x \in \mathbf{C} \cap \Lambda$. But $x \in W^s(z_{n_0}) \subseteq$

$W^s(\mathcal{A})$ and it yields to a contradiction because there is no points in $W^s(\mathcal{A}) \cap \Lambda$. Then y' and y'' coincide, and S is continuous at y. $\square$

Corollary 3.2. *S (or its continuous extension): $\mathcal{A} \to \Lambda$ is onto.*

Proof. We have that

$$\Lambda = \overline{S(R)} \subseteq \overline{S(\mathcal{A})} = S(\mathcal{A})$$

The last equality holds since S is continuous, then $S(\mathcal{A})$ is a compact set. $\square$

4 Existence of a repelling topological hypersurface

Proposition 4.1. *Λ is a topological hypersurface.*

Proof. Since S (or its extension) is onto, for every k-periodic point $y \in \Lambda$ there exists $x \in \mathcal{A}$, k-periodic such that $S(x) = y$. Let us denote $\Gamma(y) = S(W^u(x))$.

Since $W^u(x) \subset \mathcal{A}$, then $\Gamma(y) \subset \Lambda$, and $\Gamma(y)$ is a curve in $\mathcal{F}^{cu}(y) = W^u(y)$. The curve is f^k invariant and y belongs to it.

We claim that any point in $W^u(y) \cap \Lambda$ has to be in $\Gamma(y)$, if y is a periodic point in Λ.

Let $r \in \Gamma(y)$. Since $W^s(y)$ is dense in Λ, there exists $z \in W^u(y) \cap W^s(y)$ such that $d^u(z, r) < \epsilon/2$ where d^u is the restriction of the Riemannian metric of M to the leaves of $\mathcal{F}^u$ and ϵ verifies that $\cup_{x \in \Lambda} W^u_\epsilon(x) \subset W^u(\Lambda)$.
Suppose that $z \notin \Gamma(y)$, then there exists $q \in W^c(z) \cap \Gamma(y)$ such that $d^c(z, q) < \epsilon$. Since $z \in W^s(y)$ there exist $(n_j)_{j \in \mathbb{N}}$ such that

$$\lim_{j \to \infty} n_j = \infty \quad \text{and} \quad \lim_{j \to \infty} f^{n_j}(z) = y.$$

Since

$$\lim_{j \to \infty} f^{n_j}(C^z_q) \subseteq F^c(y), \quad \text{and} \quad \lim_{j \to \infty} f^{n_j}(C^z_q)$$

is not included in $W^u(\Lambda)$, there exists $y' \in \partial(W^u(\Lambda)) \cap W^c(y)$ such that $\Sigma(y) = y'$, therefore there exists y'' close to y', such that $\Sigma(f^{n_j}(z)) = y''$ with $y'' \in C^{f^{n_j}(z)}_{f^{n_j}(q)}$ because y is a continuity point of Σ.
Then $f^{-n_j}(y'') \in C^z_q \cap \partial(W^u(\Lambda))$, but $d^c(f^{-n_j}(y''), q) < \epsilon$ so $f^{-n_j}(y'')$ must

be in $W^u(\Lambda)$ which is a contradiction.
We have proved that if $U_\epsilon = \cup_{r\in\Gamma(y)} W^c_\epsilon(r)$ then

$$U_\epsilon \cap W^s(y) \subset \Gamma(y). \tag{1}$$

Suppose that there exists $w \in W^u(y)$ such that $w \notin \Gamma(y)$. Then there exists $n \in \mathbb{N}$ such that $f^{-n}(w) \in U_\epsilon \backslash \Gamma(y)$. Besides, there exists δ such that $B(f^{-n}(w), \delta) \cap \Gamma(y) = \emptyset$, and $B(f^{-n}(w), \delta) \cap W^u(y) \subset U_\epsilon$, but there is no point of $W^s(y)$ in $B(f^{-n}(w), \delta)$ by (1), which contradicts the density of $W^s(y)$. Then, we have proved that all the points in $\Lambda \cap W^u(y)$ must be in $\Gamma(y)$.

For all $x \in \Lambda$ there exists a periodic point $z \in \Lambda$ close to x. Let

$$\Gamma(x) = (\cup_{w\in\Gamma(z)} W^s_{loc}(w)) \cap W^u(x).$$

We have that $\Gamma(x)$ is a curve in $W^u(x) \cap \Lambda$. We claim that every point of $W^u(x) \cap \Lambda$ has to be in $\Gamma(x)$.

Suppose, contrary to our claim that there were a point $v \in \Lambda \cap W^u(x)$, such that $v \notin \Gamma(x)$ then $\tilde{v} = W^s_{loc}(v) \cap W^u(z)$ would be a point in $\Lambda \cap W^u(z)$, such that $\tilde{v} \notin \Gamma(z)$, which is impossible.

We have proved that $\forall x \in \Lambda$ there is a unique curve $\Gamma_x \subset W^u(x) \cap \Lambda$. Then $D_x = \cup_{z\in\Gamma_x} W^s_{loc}(z)$ is a local hypersurface of Λ. Let $V_\epsilon = \cup_{r\in D_x} W^c_\epsilon(r)$, then $V_\epsilon \cap \Lambda$ must be included in the local hypersurface D_x.

Hence Λ is a topological hypersurface. □

5 End of the proof of the Theorem

Proposition 5.1. *The Anosov flow ϕ is conjugated to a suspension.*

Proof. The topological hypersurface Λ is compact, f-invariant and $f|_\Lambda$ is hyperbolic. If $x \in \Lambda$, $f(x) \in \Lambda$ then there exists $z \in W^c(x)$ such that $z \in \Lambda$, and $C^x_z \cap \Lambda = \emptyset$.

By the version of Corollary 1.1 for repeller sets $\{F^c_f(x)\}_{x\in\Lambda}$ is topologically transversal to Λ.

Recall that as f is C^1 close to f_1, where $f_1(x) = \phi(x, 1)$ there exists a homeomorphism $h : M \to M$ close to the identity such that $h(x) = x'$, and $F^c_f(x')$ is C^1-close to $F^c_{f_1}(x)$ in compact sets. Moreover

$$h(F^c_{f_1}(x)) = F^c_f(x').$$

Since $h^{-1}(\Lambda)$ is a topological hypersurface we have that $\{F^c_{f_1}(x)\}_{x\in h^{-1}(\Lambda)}$ is topologically transversal to $h^{-1}(\Lambda)$, i.e. $\forall x \in M$ there exists $T > 0$ such that $\phi(x, T) \cap h^{-1}(\Lambda)$ "transversally".

Then ϕ, may be reparametrized in such a way that it becomes a suspension, i.e. the Anosov flow is conjugated to a suspension which is an Anosov flow, too.

Remark 5.1. The flow ϕ is conjugate to a suspension of an Anosov diffeomorphism and the hypersurface Λ is homeomorphic to the torus T^{n-1}.

We have that $f|\Lambda$ is a hyperbolic diffeomorphism. If Λ were a smooth manifold, $f|\Lambda$ would be an Anosov codimension one diffeomorphism and we could apply Frank´s result to conclude that $f|\Lambda$ is topologically conjugated to a hyperbolic toral automorphism (See [4]). Although Λ is just a topological manifold, the Frank´s proof remains valid but, in this case we need to use a C^0 version of the classical theorem of Haefliger. This can be found in Chapter 7 of [6].

Let $A : T^{n-1} \to T^{n-1}$ be an Anosov diffeomorphism such that $f|\Lambda$ is conjugated to $A|T^{n-1}$, then if ψ is the suspension of A, ϕ is conjugated to ψ. Hence the flow ϕ is conjugated to a suspension of an Anosov diffeomorphism.

The above observation completes the proof of the Theorem.

Let M a riemannian, compact surface with negative curvature. It is well known that geodesic flows can not be conjugated to a suspension flow. Then Corollary 1 holds.

References

[1] Anosov D. *Geodesic flows on Riemann manifolds with negative curvature.* Proceedings of the Steklov Institute of Mathematics, vol **90** (1967).

[2] Bonatti C. and Diaz L. *Persistent nonhypebolic transitive diffeomorphisms.* Annals of Mathematics, vol **143** (1995).

[3] Cartan E. *Lecons sur la géométrie des espaces de Riemann.* Gautthier-Villars (1928).

[4] Franks J. *Anosov diffeomorphisms.* Proceedings of the Symposium in Pure Math., vol **14** (1970).

[5] Franks J. and Williams R. *Anomalous Anosov Flows.* Global theory and dynamical systems, SLN 819, Springer-Verlag (1980).

[6] Hector G. and Hirsch U. *Introduction to the Geometry of Foliations.* Fiedr. Vieweg & Sohn, Braunschweig (1981).

[7] Hirsch M., Pugh C. and Shub M. *Invariant manifolds.* Lecture Notes in Mathematics, Springer-Verlag, vol **583** (1977).

[8] Katok A. and Hasselblat B. *Introduction to the Modern Theory of Dynamical Systems.* Cambridge University Press. (1995).

[9] Palis J. and Pugh C. *Fifty problems in dynamical systems.* Lecture Notes in Mathematics, Springer-Verlag, vol **468** (1975).

[10] Verjovsky A. *Codimension one Anosov flows.* Bol. Soc. Mat. Mexicana, vol **19** (1974).

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